Comment on 'An extreme critical space-time: echoing and black-hole perturbations'

Gérard Clément *
Laboratoire de Physique Théorique LAPTH (CNRS),
B.P.110, F-74941 Annecy-le-Vieux cedex, France

Sean A. Hayward †
Department of Science Education, Ewha Womans University,
Seodaemun-gu, Seoul 120-750, Korea

8 August 2001

Abstract

We show that the black hole perturbations of the Hayward static solution to the massless Einstein-Klein-Gordon equations are actually gauge artifacts resulting from the linearization of a coordinate transformation.

In a recent paper [1], Hayward discussed a homothetic, static solution to the massless Einstein-Klein-Gordon equations. With the spherically symmetric ansatz

$$ds^2 = r^2 d\Omega^2 - 2e^{2\gamma} dx^+ dx^- \tag{1}$$

(where $d\Omega^2$ is the line-element of the unit sphere and (r, γ) are functions of the null coordinates $x^\pm$), this very simple solution reads

$$r = (-x^+ x^-)^{1/2} \qquad \gamma = 0 \qquad \phi = \frac{1}{2}\ln(-x^+/x^-) \tag{2}$$

(where ϕ is the scalar field). Another form of the solution (2), showing explicitly its static and homothetic character, is

$$ds^2 = e^{2\rho}(d\Omega^2 + 2d\rho^2 - 2d\tau^2). \tag{3}$$

*Email: gclement@lapp.in2p3.fr
†Email: hayward@mm.ewha.ac.kr

The relation between the two coordinate systems is

$$x^{\pm} = \pm e^{\rho \pm \tau}. \tag{4}$$

Hayward argued that this solution is critical, in the sense that it lies at the threshold between black holes and naked singularities. This argument was partly based on the analysis of the linear perturbations of the static solution (3). For a given mode with complex frequency k, these linear perturbations are of the form

$$r = e^{\rho}(1 + \epsilon \tilde{r}(\tau) e^{-k\rho}) \tag{5}$$
$$\gamma = \epsilon \tilde{\gamma}(\tau) e^{-k\rho} \tag{6}$$
$$\phi = \tau + \epsilon \tilde{\phi}(\tau) e^{-k\rho}, \tag{7}$$

where ϵ is the perturbation parameter. The resulting linearized field equations reduce to a single fourth-order ordinary differential equation. Imposing weak boundary conditions, Hayward found the general solution

$$\tilde{r} = A_+ e^{\omega_A \tau} + A_- e^{-\omega_A \tau} + B_+ e^{\omega_B \tau} + B_- e^{-\omega_B \tau} \tag{8}$$

depending on four integration constants $A_\pm$, $B_\pm$, with the exponents

$$\omega_A = i\sqrt{(\Im k)^2 - 3} \qquad (\Re k = 1) \tag{9}$$
$$\omega_B = k \qquad (\Re k \leq 1/2). \tag{10}$$

However, as we now show, all the ω_B modes are spurious, and may be generated from the static solution (2) by the coordinate transformation

$$x^+ = \hat{x}^+ + 2\epsilon B_- (\hat{x}^+)^{1-k} \tag{11}$$
$$-x^- = -\hat{x}^- + 2\epsilon B_+ (-\hat{x}^-)^{1-k}, \tag{12}$$

preserving the double-null ansatz (1). To first order in ϵ, this leads to

$$r = (-x^+ x^-)^{1/2} = (-\hat{x}^+ \hat{x}^-)^{1/2}(1 + \epsilon B_- (\hat{x}^+)^{-k} + \epsilon B_+ (-\hat{x}^-)^{-k}). \tag{13}$$

Dropping the hats, this is of the form (5), with

$$\tilde{r} = B_+ e^{k\tau} + B_- e^{-k\tau}, \tag{14}$$

corresponding to the ω_B modes of (8). The perturbative black holes found in [1] are therefore simply artifacts resulting from the linearization of a coordinate transformation. To see how this happens, note from (2) that

the centre $r = 0$ occurs at the null lines $x^{\pm} = 0$, whereas according to the linearized expression (13), there is a centre $r = 0$ at

$$B_-(\hat{x}^+)^{-k} + B_+(-\hat{x}^-)^{-k} = -1/\epsilon \tag{15}$$

which, for $B_+ < 0$, $B_- > 0$, is a spatial curve. This spatial centre may now be seen to be an artifact produced by dropping the ϵ^2 term in (13). Similar remarks apply to the linearized trapping horizons $\partial_\pm r = 0$, which do not appear if the invariant definition $e^{-2\gamma}\partial_+ r\partial_- r = 0$ is used.

The problem may be regarded as due to the weak nature of the boundary conditions imposed in [1]. Imposing stronger boundary conditions analogous to those in the perturbative analysis by Frolov [2] of the related critical Roberts solution [3], for instance that $\tilde{r}$ is bounded as $\tau \to \pm\infty$, these spurious gauge modes may be excluded.

Note that this has not disproved the argument that the static solution lies at the threshold between black holes and naked singularities. This still seems likely, since a future-null singularity is just on the verge of being naked, and a marginally future-trapped singularity is just on the verge of disappearing inside a black hole. However, such solutions apparently cannot be found by linear perturbations.

The conclusion is that the only genuine perturbative modes of the Hayward solution are the ω_A modes in (8). As discussed in [1], the imaginary part of the frequency ($\Im k \geq \sqrt{3}$) of these modes produces echoing reminiscent of the numerical simulations of Choptuik [4].

References

[1] S.A. Hayward, *Class. Quantum Grav.* **17**, 4021 (2000)

[2] A.V. Frolov, *Phys. Rev.* D **56**, 6433 (1997)

[3] M.D. Roberts, *Gen. Rel. Grav.* **21**, 907 (1989); P.R. Brady, *Class. Quantum Grav.* **11**, 1255 (1994); Y. Oshiro, K. Nakamura and A. Tomimatsu, *Progr. Theor. Phys.* **91**, 1265 (1994)

[4] M.W. Choptuik, *Phys. Rev. Lett.* **70**, 9 (1993)